Multimode quantum optical metrology
Gaussian systems for quantum enhanced phase estimation

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Introduction and motivation

The problem: we want to read the pixels of an image. The unknown phases label each pixel.

- The unknown phases are relative to some reference phase: we cannot measure absolute phases.
- How the reference mode should be defined?
- Is it better to estimate the phases simultaneously or individually (with mean energy restriction)?

The performance of any estimation process is captured by the covariance matrix of the estimator $V(\varphi)$.

Lower bound: $V(\varphi) \geq H^{-1}$

Bound of the total variance of all the parameters:

$Tr(V(\varphi)) \geq Tr(H^{-1})$
Reference mode: example

\[ |\beta\rangle \quad \rightarrow \quad |0\rangle \quad \rightarrow \quad 1/2 \quad \rightarrow \quad |1\rangle \quad \rightarrow \quad \phi_1 \]

\[ |0; |\xi|\rangle \quad \rightarrow \quad (0) \quad \rightarrow \quad 1/2 \quad \rightarrow \quad \phi_1 \]

\[ \hat{U}_\phi = e^{i\phi_1 \hat{n}_1} \]

\[ H = (1 + e^{2|\xi|})|\beta|^2 + (2 + \cosh 2|\xi|) \sinh^2 |\xi| \]

wrong

\[ \hat{U}_\phi = \hat{U}_{\phi_0} e^{i\phi_0 \hat{n}_0 + i\phi_1 \hat{n}_1} = e^{i\phi_1 (\hat{n}_1 - \hat{n}_0)} \]

Unmeasurable phase

\[ H = 4(|\beta|^2 e^{2\xi} + \sinh^2 |\xi|) \]

correct
Reference mode and SU(n) algebra

By setting the reference mode’s phase to zero may lead to wrong results. The proper way is to keep the whole problem in SU(n)

\[ \hat{U}_\phi = \hat{U}'_\phi \exp(-i\varphi \hat{n}_0) = \exp(i\phi_1(\hat{n}_1 - \hat{n}_0) + \ldots + i\phi_d(\hat{n}_d - \hat{n}_0)) = \exp \left( i \sum_{i=1}^{d} \phi_i (\hat{n}_i - \hat{n}_0) \right) \]

\[ \hat{g}_i \]

Captures an unmeasurable overall phase

\[ \varphi = \phi_0 + \ldots + \phi_d \]
General pure Gaussian input and the comparison we intend to do

\[ H_{i,j} = 4(\langle \hat{g}_i \hat{g}_j \rangle - \langle \hat{g}_i \rangle \langle \hat{g}_j \rangle) = 4(h_{i,j} - h_{i,0} - h_{0,j} + h_{0,0}) \]

\[ G(\mu) = \exp \left[ \frac{\mu^T M^{-1} \mu + \mu^T M^{-1} \mu^*}{4} \right] \]

\[ M^{-1} = 2 \begin{pmatrix} \mathbf{E} & \mathbf{E}^T \\ -\mathbf{N}^\dagger \mathbf{E} & \mathbf{E}^T \end{pmatrix} \]

Derivatives wr.t. to \( \mu \)

\[ h_{i,j} = 4 \left( (\mathbf{E} \mathbf{N} - \mathbf{\gamma} \mathbf{\gamma}^T) \circ (\mathbf{E} \mathbf{N} - \mathbf{\gamma} \mathbf{\gamma}^T)^* - (\mathbf{\gamma} \mathbf{\gamma}^T) \circ (\mathbf{\gamma} \mathbf{\gamma}^T)^* + \frac{1}{4} (\mathbf{E} + \mathbf{E}^*) \circ (\mathbf{E} + \mathbf{E}^* + 2 \mathbf{\gamma} \mathbf{\gamma}^T + 2 \mathbf{\gamma}^T \mathbf{\gamma}^T) - (\mathbf{E} + \mathbf{\gamma} \mathbf{\gamma}^T) \circ \mathbf{I} \right)_{i,j}, \]

\[ R = \frac{\text{Tr}(\mathbf{H}_{\text{sim}}^{-1})}{\text{Tr}(\mathbf{H}_{\text{ind}}^{-1})} \]
Assumptions and optimization

**Assumptions:** Orthogonal transformation and same squeezing

**Optimization (Lagrange multipliers):** All energy into squeezing

Comparison

$$R = \frac{\text{Tr}(H_{\text{sim}}^{-1})}{\text{Tr}(H_{\text{ind}}^{-1})} = 1 - \frac{d-1}{2d} \tanh^2 |\xi| \leq 1$$

Enhancement because:
- utilizing squeezed states
- one reference mode in the simultaneous strategy

$$R \geq \frac{1}{2} \lim_{d \to \infty} \lim_{|\xi| \to \infty} R = \frac{1}{2}$$

The ratio doesn't go to zero because Gaussian states are not the best for estimation (even though very practical) and/or because of the assumptions.

$$R_{\text{lim}} = \lim_{d \to \infty} R = 1 - \frac{1}{2} \tanh^2 |\xi|$$

For (almost) realistic squeezing the lower bound can be achieved
Coherent light and squeezed vacua

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \cdots \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

\[
\beta \in \mathbb{R}
\]

\[
\xi = -|\xi|
\]
Comparison

Note however that the all-squeezed scenario performs better.

\[ R = 1 - \frac{(d-1) \cosh 2|\xi| \sinh^2 |\xi|}{\beta^2 e^2 |\xi| + \sinh^2 |\xi| + \frac{d-1}{2} \sinh^2 2|\xi|} \leq 1 \]

Goes to zero for large squeezing.

\[ R_{lim} = \frac{1}{2 \cosh^2 |\xi|} \]
Attainability of the quantum limit

For multiple parameters, a sufficient condition (and necessary condition using asymptotic normality) for the saturation is:

$$\text{Tr} \left( \hat{\rho}_\phi \left[ \hat{L}_i, \hat{L}_j \right] \right) = 0$$

For quantum estimation using pure states, the multi-parameter QCRB can be saturated without resorting to local asymptotic normality.

Our case:

$$\frac{\partial \hat{\rho}_\phi}{\partial \phi_i} = i \left[ (\hat{n}_i - \hat{n}_0), \hat{\rho}_\phi \right] \quad \xrightarrow{\hat{L}_i = 2i \left[ (\hat{n}_i - \hat{n}_0), \hat{\rho}_\phi \right]} \quad \text{Tr} \left( \hat{\rho}_\phi \left[ \hat{L}_i, \hat{L}_j \right] \right) = 0$$

$$\hat{L}_i : \text{SLDs}$$

$$\frac{\partial \hat{\rho}_\phi}{\partial \phi_i} = \frac{\hat{L}_i \hat{\rho}_\phi + \hat{\rho}_\phi \hat{L}_i}{2}$$

$$H_{i,j} = \frac{1}{2} \text{Tr} \left[ \hat{\rho}_\phi \left( \hat{L}_i \hat{L}_j + \hat{L}_j \hat{L}_i \right) \right]$$
**Simultaneous better than individual:** one reference mode, more energy (more squeezing = larger variance) in the sensing modes

In the multiple phase estimation **modal entanglement is not always helpful**, in the sense of comparison simultaneous/individual estimation.

Gaussian states appear to have some limitations compared to entangled states (e.g. GNS states).

Frequency squeezed state and gravitational waves
Thank you

arxiv.org/abs/1605.04819