Multimode quantum optical metrology

Gaussian systems for quantum enhanced phase estimation

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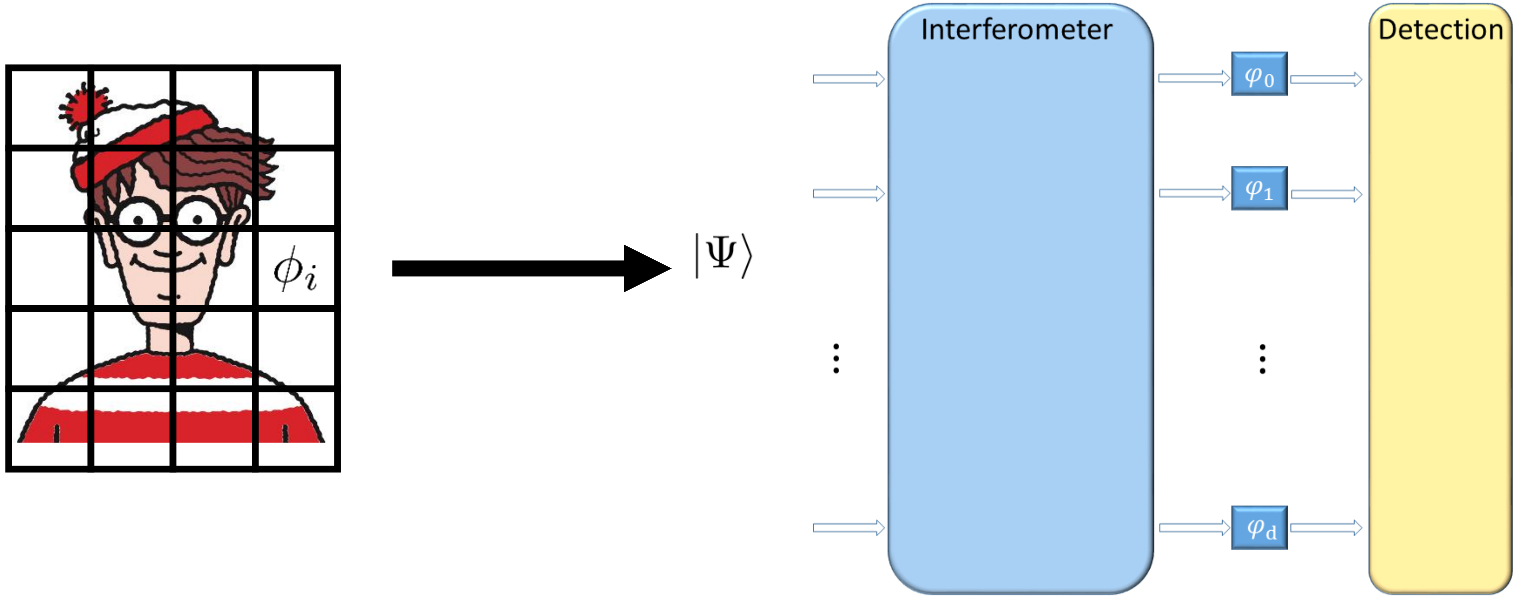
The problem: we want to read the pixels of an image. The unknown phases label each pixel.



The performance of any estimation process is captured by the covariance matrix of the estimator $V(\boldsymbol{\varphi})$.

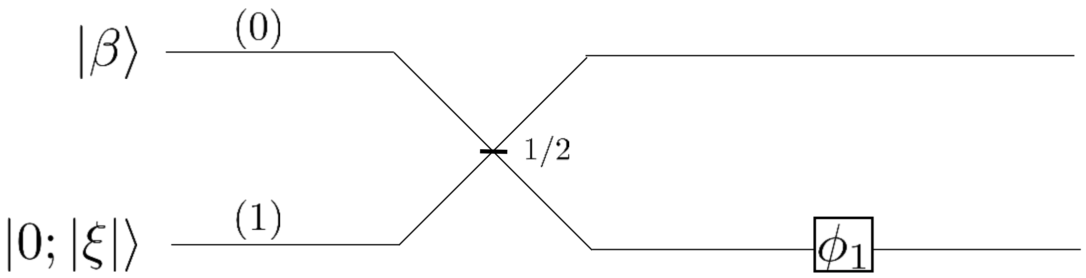
Lower bound: $V(\boldsymbol{\varphi}) \geq \mathrm{H}^{-1}$

Bound of the total variance of all the parameters:

$$Tr(V(\boldsymbol{\varphi})) \geq Tr(\mathrm{H}^{-1})$$

- The unknown phases are relative to some reference phase: we cannot measure absolute phases.
- How the reference mode should be defined?
- Is it better to estimate the phases simultaneously or individually (with mean energy restriction)?

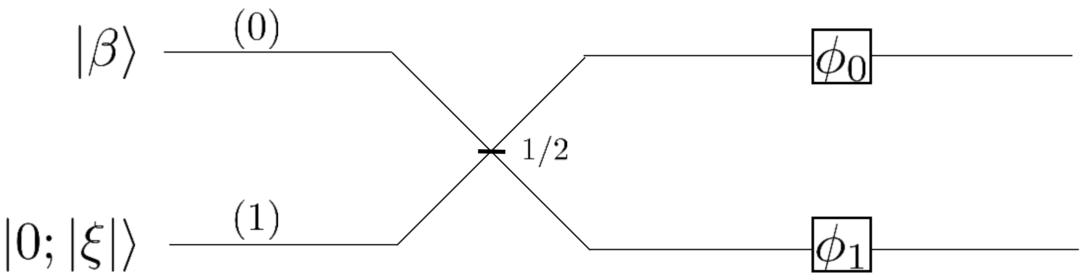
Reference mode: example



$$\hat{\mathcal{U}}_\phi = e^{i\phi_1 \hat{n}_1}$$

$$H = (1 + e^{2|\xi|})|\beta|^2 + (2 + \cosh 2|\xi|) \sinh^2 |\xi|$$

wrong

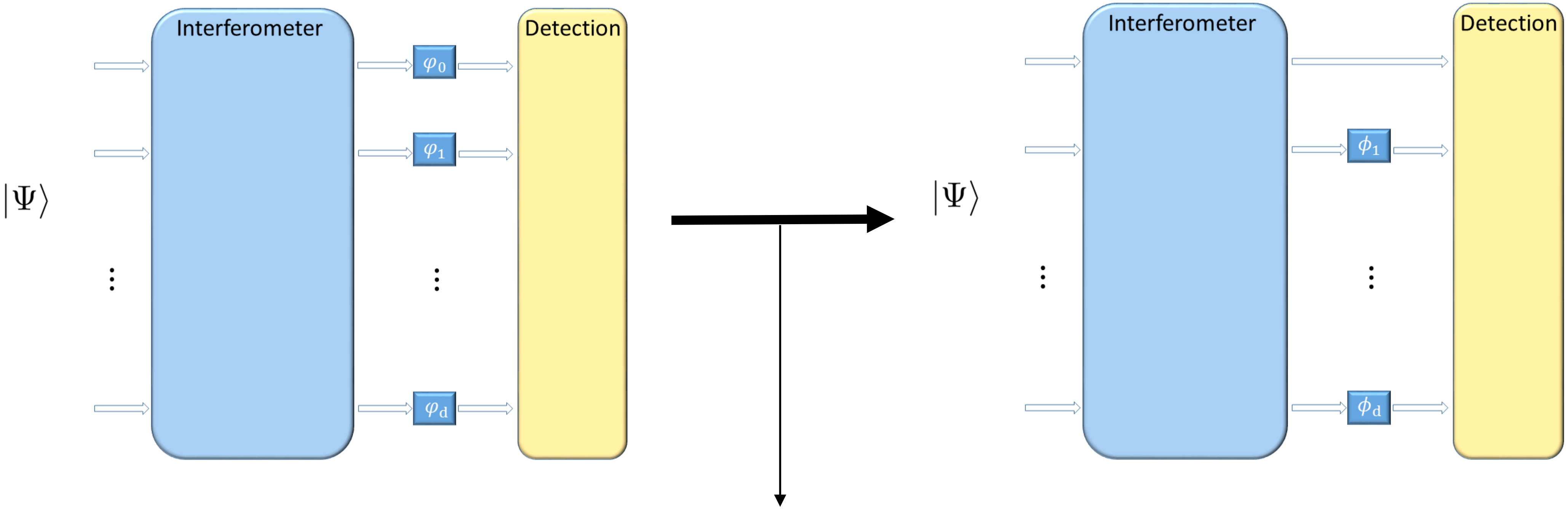


$$\hat{\mathcal{U}}_\phi = e^{-i\underbrace{(\phi_0+\phi_1)}_{\text{Unmeasurable phase}}\hat{n}_0} e^{i\phi_0\hat{n}_0 + i\phi_1\hat{n}_1} = e^{i\phi_1(\hat{n}_1 - \hat{n}_0)}$$

$$H = 4(|\beta|^2 e^{2\xi} + \sinh^2 |\xi|)$$

correct

Reference mode and SU(n) algebra



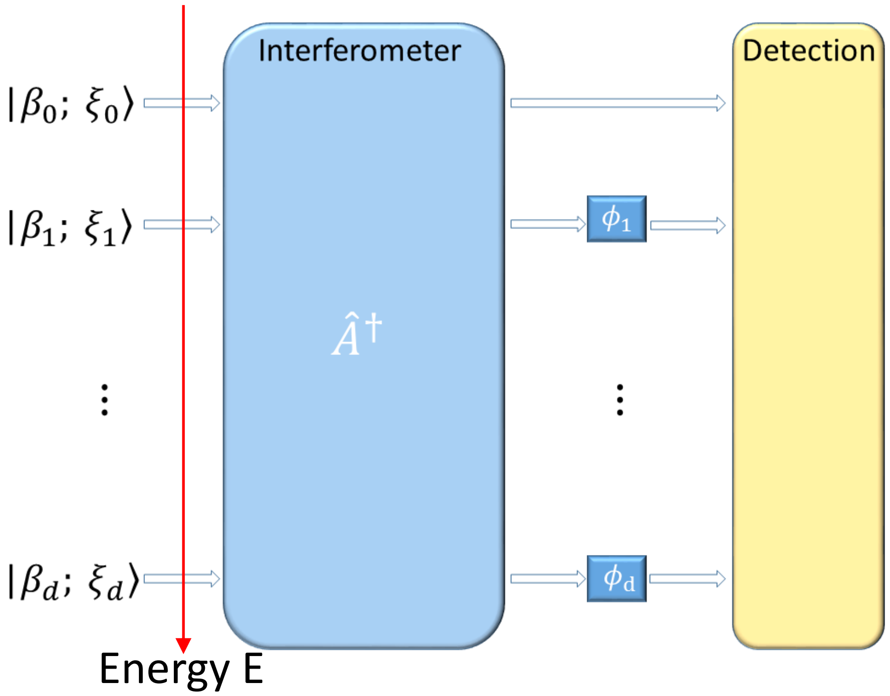
$$\hat{\mathcal{U}}_{\boldsymbol{\phi}} = \hat{\mathcal{U}}'_{\boldsymbol{\phi}} \exp(-i\varphi\hat{n}_0) = \exp(i\phi_1(\hat{n}_1 - \hat{n}_0) + \ldots + i\phi_d(\hat{n}_d - \hat{n}_0)) = \exp\left(i \sum_{i=1}^{d} \phi_i \underbrace{(\hat{n}_i - \hat{n}_0)}_{\hat{g}_i}\right)$$

By setting the reference mode's phase to zero may lead to wrong results. The proper way is to keep the whole problem in SU(n)

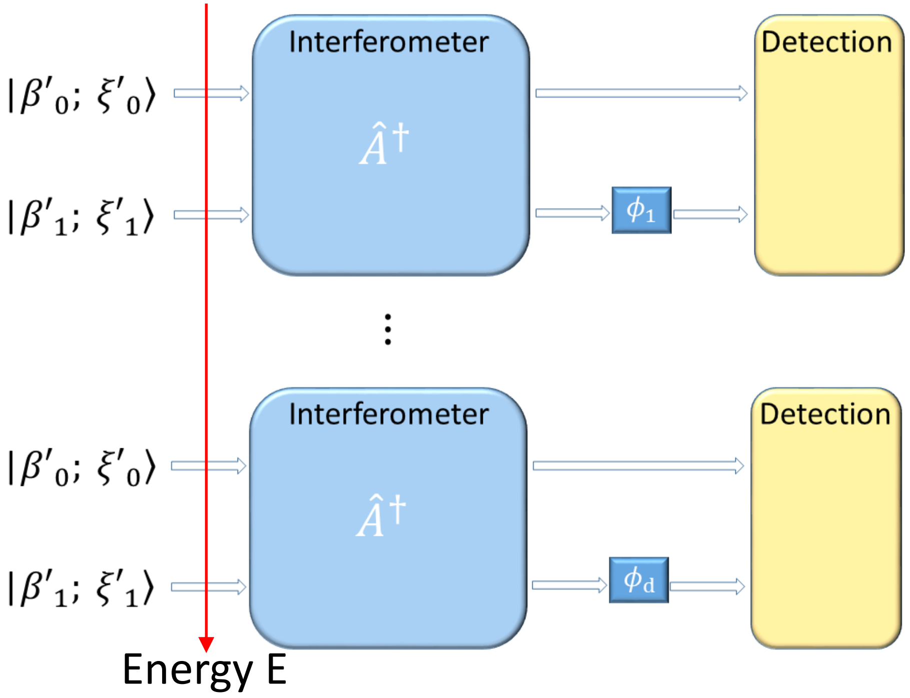
$$\varphi = \phi_0 + \ldots + \phi_d$$

Captures an unmeasurable overall phase

General pure Gaussian input and the comparison we intend to do



$$R = \frac{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{sim}}^{-1}\right)}{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{ind}}^{-1}\right)}$$



$$H_{i,j} = 4(\langle \hat{g}_i \hat{g}_j \rangle - \langle \hat{g}_i \rangle \langle \hat{g}_j \rangle) = 4(h_{i,j} - h_{i,0} - h_{0,j} + h_{0,0})$$

$$G(\boldsymbol{\mu}) = \exp\left[\frac{\boldsymbol{r}_b^\dagger \mathbf{M}^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^\dagger \mathbf{M}^{-1}\boldsymbol{r}_b + \boldsymbol{\mu}^\dagger \mathbf{M}^{-1}\boldsymbol{\mu}}{4}\right] \qquad \mathbf{M}^{-1} = 2\begin{pmatrix} \mathbf{E} & -\mathbf{N}\mathbf{E}^T \\ -\mathbf{N}^\dagger \mathbf{E} & \mathbf{E}^T \end{pmatrix}$$

$$\boldsymbol{r}_b = (b_0, \ldots, b_d, b_0^*, \ldots, b_d^*)^T \equiv (\boldsymbol{b}, \boldsymbol{b}^*)^T$$

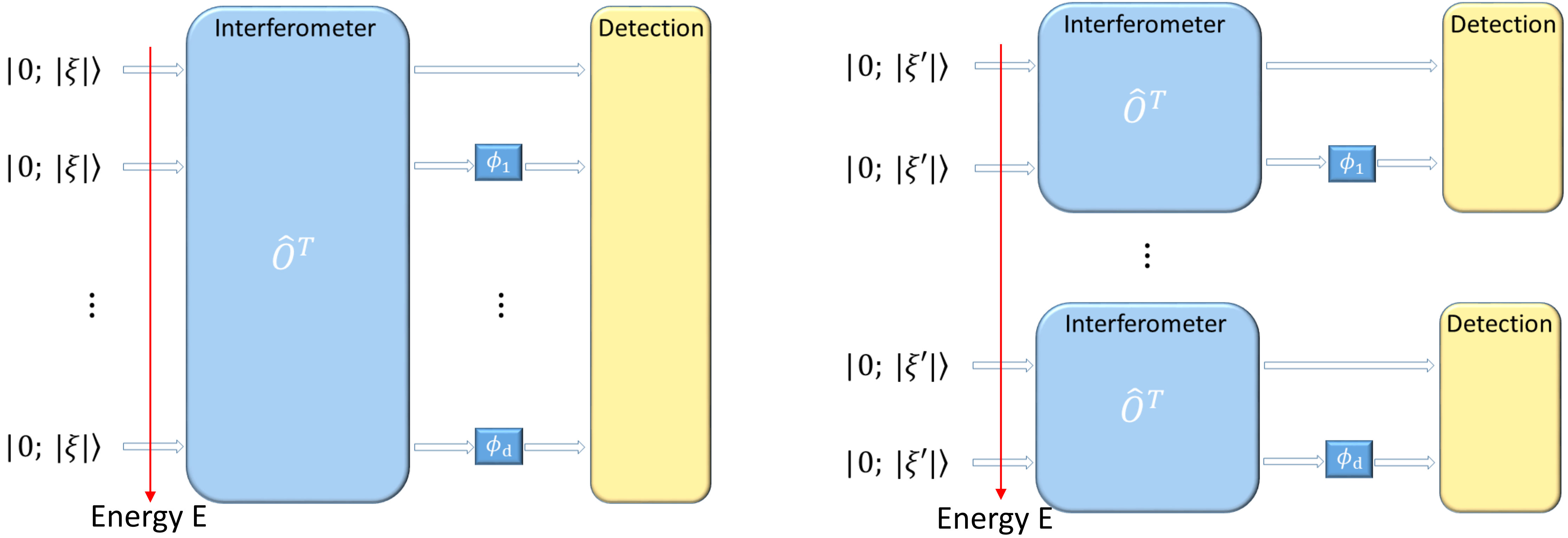
$$b_j = \sum_{k=0}^{d} A_{kj}^* \left(\beta_k + \beta_k^* \tanh|\xi_k|\right)$$

$$\boldsymbol{\mu} = (\lambda_0, \ldots, \lambda_d, \lambda_0^*, \ldots, \lambda_d^*)^T$$

Derivatives wr.t. to mu

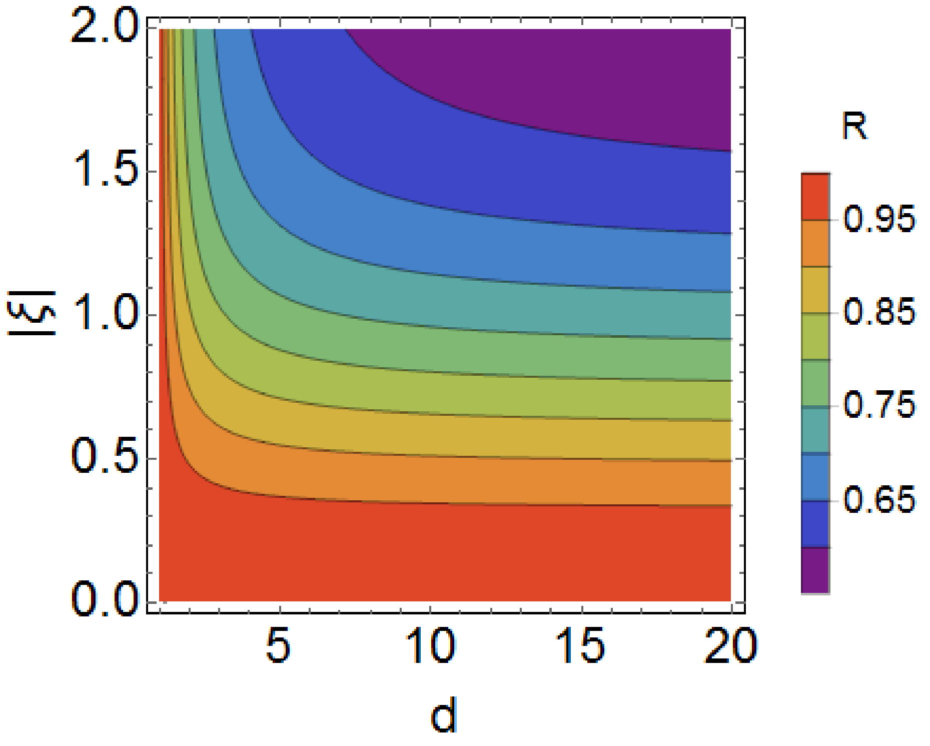
$$h_{i,j} = 4\Big((\mathbf{E}\mathbf{N} - \boldsymbol{\gamma}\boldsymbol{\gamma}^T) \circ (\mathbf{E}\mathbf{N} - \boldsymbol{\gamma}\boldsymbol{\gamma}^T)^* - (\boldsymbol{\gamma}\boldsymbol{\gamma}^T) \circ (\boldsymbol{\gamma}\boldsymbol{\gamma}^T)^* + \tfrac{1}{4}(\mathbf{E} + \mathbf{E}^*) \circ (\mathbf{E} + \mathbf{E}^* + 2\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger + 2\boldsymbol{\gamma}^*\boldsymbol{\gamma}^T) - (\mathbf{E} + \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger) \circ \mathbf{I}\Big)_{i,j},$$

Assumptions and optimization



Assumptions: Orthogonal transformation and same squeezing → Optimization (Lagrange multipliers): All energy into squeezing

Agrees with recent results: arxiv.org/abs/1601.05912 P. A. Knott et al.



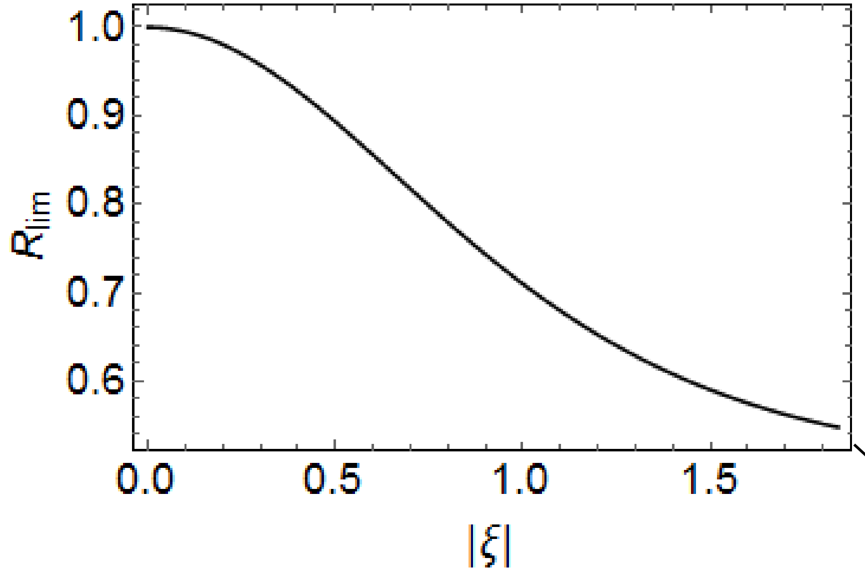
$$R = \frac{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{sim}}^{-1}\right)}{\mathrm{Tr}\left(\mathbf{H}_{\mathrm{ind}}^{-1}\right)} = 1 - \frac{d-1}{2d}\tanh^2|\xi| \leq 1$$

Enhancement because:

- utilizing squeezed states
- one reference mode in the simultaneous strategy

$$R \geq \frac{1}{2} \qquad \lim_{d\to\infty}\lim_{|\xi|\to\infty} R = \frac{1}{2}$$

The ratio doesn't go to zero because Gaussian states are not the best for estimation (even though very practical) and/or because of the assumptions.



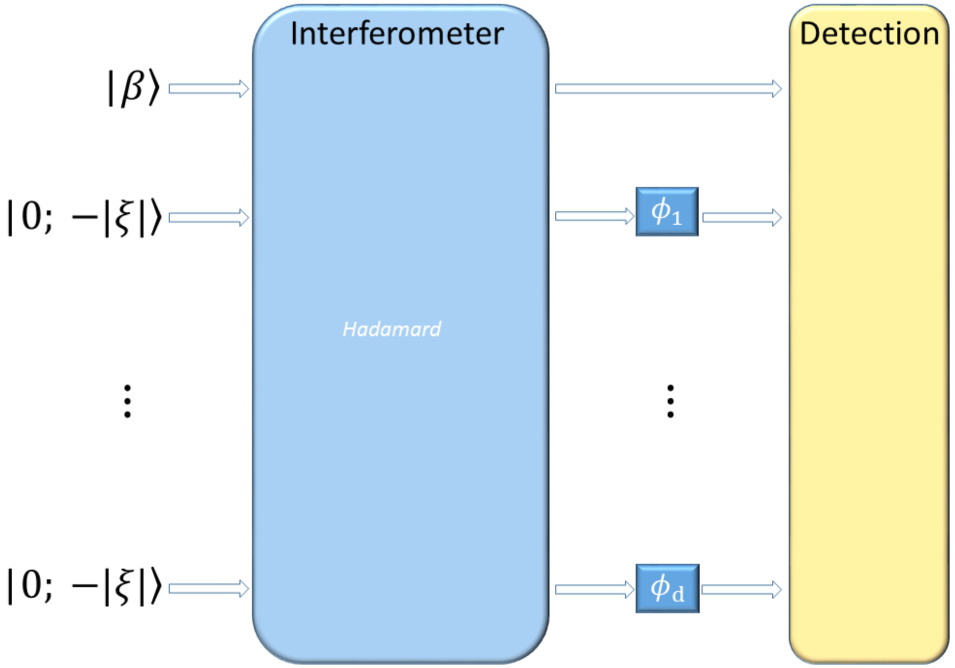
$d \to \infty$

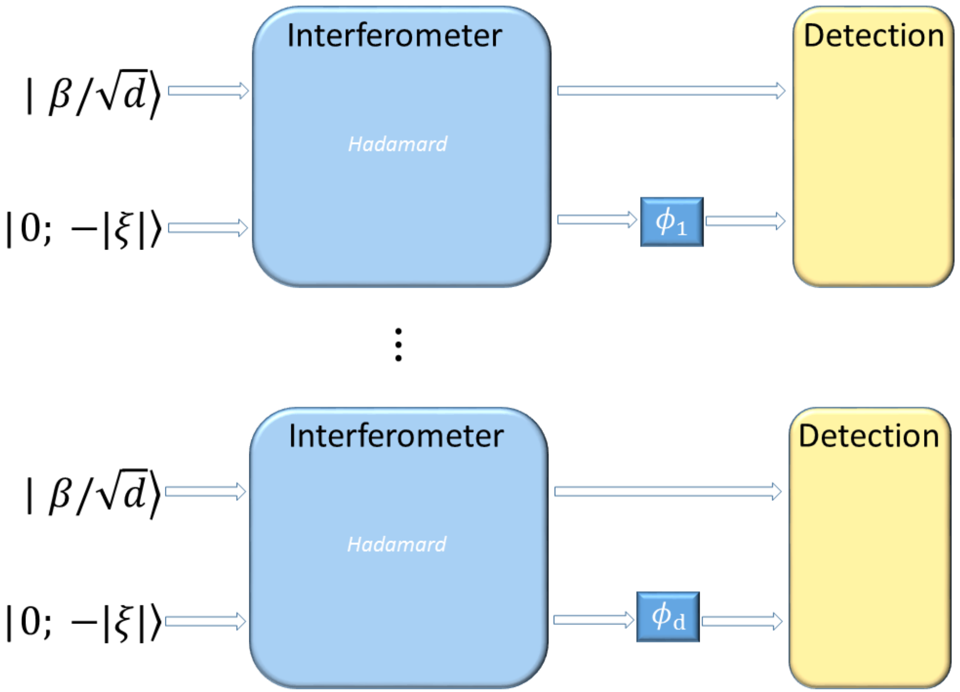
16 dB

$$R_{\mathrm{lim}} = \lim_{d\to\infty} R = 1 - \frac{1}{2}\tanh^2|\xi|$$

For (almost) realistic squeezing the lower bound can be achieved

Coherent light and squeezed vacua



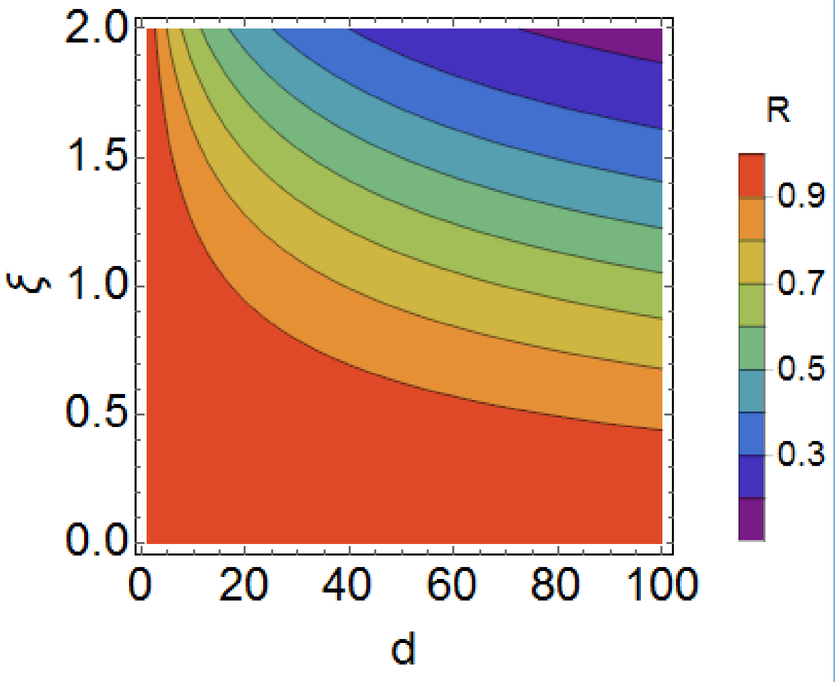


$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \ldots \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

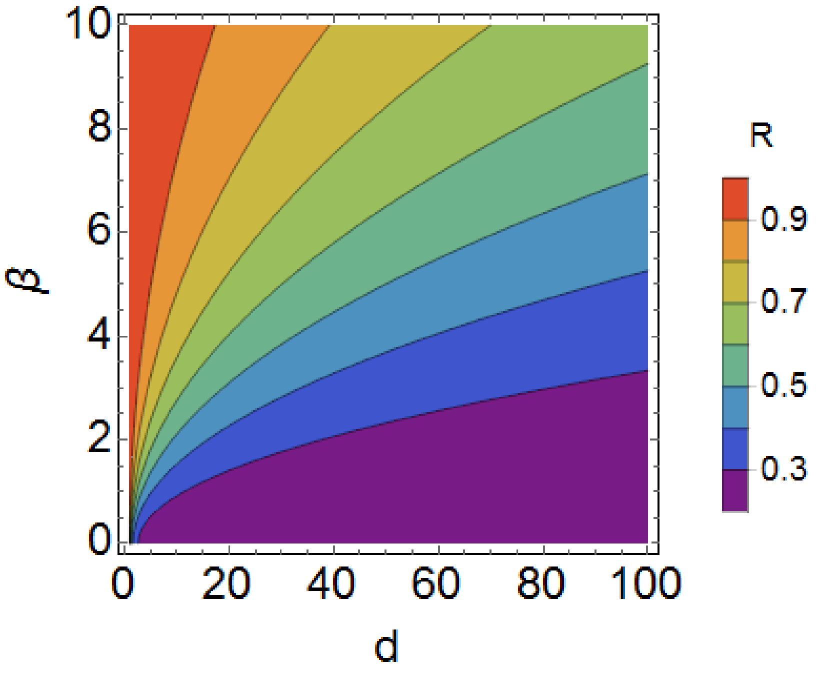
Same input energy in both strategies: the squeezing is the same in both scenarios, the energy conservation comes from the coherent amplitude.

$$\beta \in \Re$$

$$\xi = -|\xi|$$

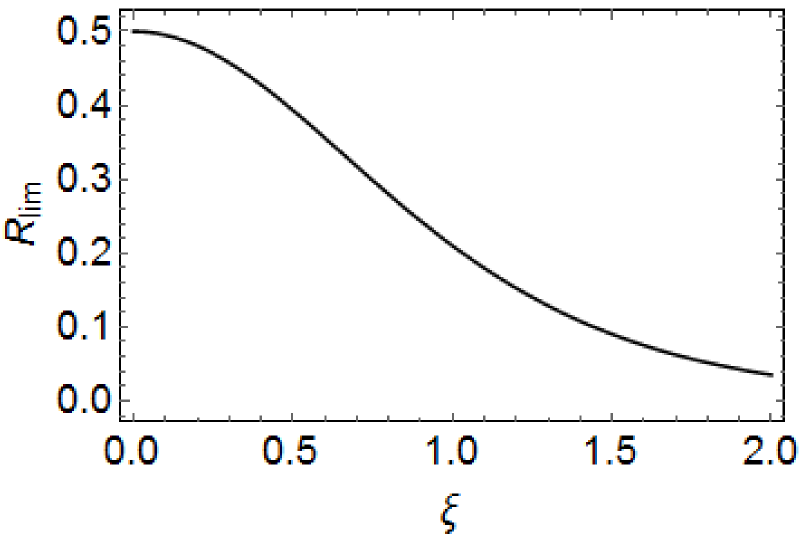


β=10



ξ=1

$$R = 1 - \frac{(d-1)\cosh 2|\xi|\sinh^2|\xi|}{\beta^2 e^{2|\xi|} + \sinh^2|\xi| + \frac{d-1}{2}\sinh^2 2|\xi|} \leq 1$$



$$d \to \infty$$

$$R_{lim} = \frac{1}{2\cosh^2|\xi|}$$

Goes to zero for large squeezing

Note however that the all-squeezed scenario preforms better

For multiple parameters, a sufficient condition (and necessary condition using asymptotic normality) for the saturation is:

$$\mathrm{Tr}\left(\hat{\rho}_\phi\left[\hat{\mathcal{L}}_i,\hat{\mathcal{L}}_j\right]\right)=0$$

$\hat{\mathcal{L}}_i$: SLDs

$$\frac{\partial\hat{\rho}_\phi}{\partial\phi_i}=\frac{\hat{\mathcal{L}}_i\hat{\rho}_\phi+\hat{\rho}_\phi\hat{\mathcal{L}}_i}{2}$$

$$\mathbf{H}_{i,j}=\frac{1}{2}\mathrm{Tr}\left[\hat{\rho}_\phi\left(\hat{\mathcal{L}}_i\hat{\mathcal{L}}_j+\hat{\mathcal{L}}_j\hat{\mathcal{L}}_i\right)\right]$$

For quantum estimation using **pure states**, the multi-parameter QCRB can be saturated without resorting to local asymptotic normality.

Our case: $\frac{\partial\hat{\rho}_\phi}{\partial\phi_i}=i\left[(\hat{n}_i-\hat{n}_0),\hat{\rho}_\phi\right]\longrightarrow\hat{\mathcal{L}}_i=2i\left[(\hat{n}_i-\hat{n}_0),\hat{\rho}_\phi\right]\longrightarrow\mathrm{Tr}\left(\hat{\rho}_\phi\left[\hat{\mathcal{L}}_i,\hat{\mathcal{L}}_j\right]\right)=0$

Simultaneous better than individual: one reference mode, more energy (more squeezing = larger variance) in the sensing modes

In the multiple phase estimation **modal entanglement is not always helpful**, in the sense of comparison simultaneous/individual estimation.

Gaussian states appear to have some limitations compared to entangled states (e.g. GNS states).

Frequency squeezed state and gravitational waves

Thank you

arxiv.org/abs/1605.04819